View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 2, 446–466 (1961)

Dynamic Programming of Some Sequential Sampling Design

MINORU SAKAGUCHI

University of Electro-Communications, Chofu, Tokyo, Japan

Submitted by Richard Bellman

I. SUMMARY

Let $\{x_n\}$ be independent random variables with a common distribution function $F(x)$. We observe the x_n sequentially and can stop at any time; if we stop with x_n we receive the payoff $f_n(x_1, \ldots, x_n)$. *Problem:* What stopping rule maximizes the expected payoff? It is shown that for

$$f_n(x_1, \ldots, x_n) = x_n - cn,$$

where $c > 0$ is the cost per unit observation, the optimum stopping rule when the first moment of the x_n exists is: Stop with the first $x_n > \alpha$ where α is the root of the equation

$$\int_{\alpha}^{\infty} (x - \alpha)\, dF(x) = c;$$

the expected payoff is then α. This result is proved in Section II.

Two directions of generalization of the problem will be given and discussed in the succeeding two sections.

A more realistic version of the problem deals with the situation where the population from which random variables are drawn has an unknown distribution function. We shall treat in Section III the case in which the distribution is normal with known variance and unknown mean.

Section IV is concerned with the problem of two populations. Here the problem is that of maximizing the expected payoff in, at most, N_1 and N_2 drawings from the populations Π_1 and Π_2, respectively, when at each drawing we are free to choose between Π_1 and Π_2. The results obtained in Section II are applied in this section to derive the optimal design of sampling.

In this paper, we shall consider all these problems by means of discussions of the functional equations derived from the corresponding decision processes. Using techniques in the theory of dynamic programming [1] we shall determine the structures of the optimal rules for the problems.

II. Optimum Stopping Rule

Let us suppose that we are playing a game in which we are allowed to make as many as N successive draws from a hypothetical population with the cumulative distribution function $F(x)$. We are allowed to stop the game at the end of any draw, and are payed an amount equal to the result of that draw minus the total cost of observations previously drawn. At each step, the decision of whether or not to continue clearly depends on (1) the value just drawn, and (2) the stage of the game. Our first statistical problem is that of finding a strategy which will maximize the expected payoff. This problem is clearly a generalization of a problem in the Sasieni-Yaspan-Friedman book [3, p. 281] and a variant of a problem considered by Robbins [2].

Using the functional equation technique of dynamic programming [1], we can easily solve this problem as follows:

Let us define the sequence of functions
$f_n(x) =$ expected payoff, obtained using an optimal rule when n stages are left in the future and starting with the observed value x just drawn at the previous stage,
for $n = 0, 1, \ldots, N - 1$.

Then it is evident that we have

$$f_0(x) = x - cN$$

$$f_{n+1}(x) = \max \begin{bmatrix} S: \ x - (N - n - 1)c \\ C: \ \int f_n(y)\, dF(y) \end{bmatrix}, \qquad n = 0, 1, \ldots, N - 2 \tag{1}$$

where the symbols S and C in the maximand at the right-hand side of the equation refer to the possible actions, to be chosen by the decision-maker, "stop sampling" and "continue sampling," respectively.

It is easy to solve these equations and we get

$$f_n(x) = \max\,(x, \mu_n - c) - (N - n)c, \qquad n = 0, 1, \ldots, N - 1 \tag{2}$$

where $\{\mu_n\}$ is the sequence of numbers defined by the recurrence relation

$$\mu_n = \int \max\,(x, \mu_{n-1} - c)\, dF(x), \qquad (n = 1, \ldots, N;\ \mu_0 = -\infty) \tag{3}$$

where the integrals are assumed to exist.

The optimal choice, when n stages are left in the future and the value x was observed at the previous stage, is made according to the following rule:

$$\begin{cases} \text{if} \quad x > \mu_n - c, & \text{stop sampling;} \\ \text{if} \quad x \leqslant \mu_n - c, & \text{continue sampling} \end{cases} \tag{4}$$

for $n = 0, 1, \ldots, N - 1$.

For $n = N$ we need another consideration, since we have no previous observation for the case. If we do not enter the game at all, the return is of course zero. If we enter the game, the expected payoff will be

$$\int f_{N-1}(y)\, dF(y) = \int \{\max(y, \mu_{N-1} - c) - c\} dF(y) = \mu_N - c$$

by (2) and (3). Thus we have a rule which determines the optimal initial choice, when any observation is not yet drawn, as follows:

$$\begin{cases} \text{if} \quad 0 > \mu_N - c, & \text{do not enter the game;} \\ \text{if} \quad 0 \leqslant \mu_N - c, & \text{take the first observation.} \end{cases} \tag{5}$$

Summarizing the above we get:

THEOREM 1. *The optimum stopping rule for the problem is determined by* (5) *and* (4). *The expected payoff obtained using the optimal rule is* $\max(\mu_N - c, 0)$.

Let us now introduce a nonnegative and nonincreasing function

$$A(x) \equiv \int \max(y - x, 0)\, dF(y) = \int_x^\infty (y - x)\, dF(y). \tag{6}$$

Then (3) can be rewritten as

$$\mu_n = A(\mu_{n-1} - c) + \mu_{n-1} - c. \tag{7}$$

We at once have the

COROLLARY 1.1. *If* $c = 0$, *the critical numbers* $\mu_1, \ldots, \mu_N$ *which characterize the optimum stopping rule of the problem are given by*

$$\mu_n = A(\mu_{n-1}) + \mu_{n-1}, \qquad n = 2, \ldots, N \tag{8}$$

$$\mu_1 = \int x\, dF(x),$$

and we have $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_N$.

If $c > 0$, an interesting result is obtained when we let $N \to \infty$.

COROLLARY 1.2. *If $c > 0$ and if we are allowed to make an infinitely many succesive draws from the population, then the optimum stopping rule is as follows:*

$$\begin{cases} \textit{If} \quad 0 > \alpha(c), & \textit{do not enter the game;} \\ \textit{if} \quad 0 \leqslant \alpha(c), & \textit{take the first observation} \end{cases}$$

and then, continuing the succesive observations, stop with the first $x_m > \alpha(c)$, where $\alpha(c)$ is the root of the equation

$$A(\alpha) \equiv \int_{\alpha}^{\infty} (x - \alpha)\, dF(x) = c. \tag{9}$$

The expected payoff is then $\max(\alpha(c), 0)$.

PROOF: The critical numbers μ_n, given by (3), which characterize the optimum stopping rule for fixed N, do not depend on the particular choice of N. Hence we have only to show that the sequence $\{\mu_n\}$ converges and that

$$\lim_{n \to \infty} \mu_n = \alpha(c) + c.$$

Let

$$B(x) \equiv A(x) + x \tag{10}$$

and

$$\nu_n = \mu_n - c$$

then (7) can be rewritten as

$$\nu_n = B(\nu_{n-1}) - c. \tag{11}$$

Now we can easily see (as in Fig. 1) the convergence of the sequence $\{\nu_n\}$ defined by (11), if we assume the strictly decreasing property of $A(x)$ near $x = \alpha(c)$. The limit of convergence is given by the root of the equation

$$x = B(x) - c$$

i.e., $A(x) = c$. If $c > 0$ and $A(x)$ is strictly decreasing for every x, then the root $\alpha(c)$ of the equation (9) exist uniquely. This completes the proof of the corollary.

Next we shall show several examples of the critical numbers μ_n or $\alpha(c)$.

(a) *Normal distribution*

$$dF(x) = \phi(x)\,dx, \qquad \phi(x) \equiv (2\pi)^{-1/2} \exp(-x^2/2)$$

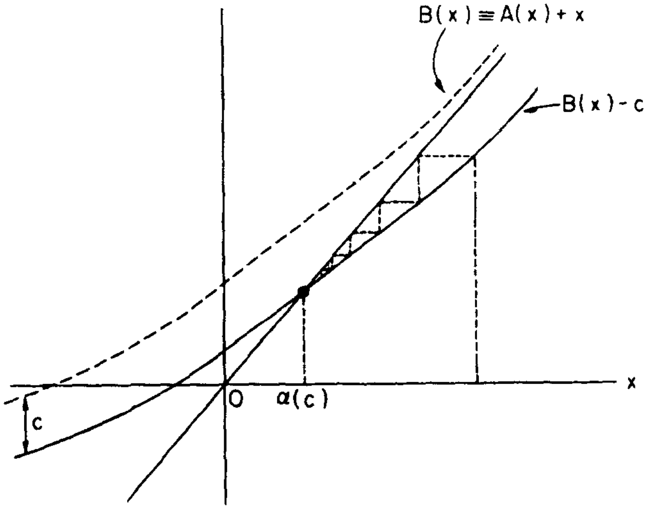

FIG. 1

We have for every x

$$A(x) = A^*(x) \equiv \int_x^\infty (y - x)\phi(y)\,dy = \phi(x) - x\Phi(x)$$

$$\left(\Phi(x) \equiv \int_x^\infty \phi(y)\,dy\right)$$

since we have

$$\int_x^\infty y\phi(y)\,dy = \phi(x)$$

for every x. Thus if $c = 0$ we get by (8) the strictly increasing sequence

$$\mu_n = \phi(\mu_{n-1}) + \mu_{n-1}(1 - \Phi(\mu_{n-1})), \qquad n = 2, 3, \ldots; \mu_1 = 0. \quad (8')$$

We obtain, for example,

n	2	3	4	5	6	7	8	9	10
μ_n	0.399	0.629	0.790	0.912	1.011	1.090	1.160	1.223	1.276

In the case of normal distribution with mean θ and variance σ^2 we can easily find that the critical numbers ν_n are given by

$$\nu_n = \sigma\mu_n + \theta$$

where μ_n are defined by (8′).

Table I gives the numerical values of $A^*(x) = \phi(x) - x\Phi(x)$ for $-2.00 < x < 3.00$.

(b) *Exponential distribution*

$$dF(x) = \beta^{-1} e^{-x/\beta}\, dx \qquad (x \geqslant 0).$$

We have for $x \geqslant 0$

$$A(x) = \int_x^\infty (y - x)\beta^{-1} e^{-y/\beta}\, dy = \beta\, e^{-x/\beta}.$$

Hence if $c = 0$ we obtain by (8) the strictly increasing sequence

$$\mu_n = \beta \exp(-\mu_{n-1}/\beta) + \mu_{n-1} \qquad (n = 2, 3, \dots; \mu_1 = \beta).$$

For example,

n	2	3	4	5	6	7	8	9	10
μ_n/β	1.368	1.623	1.821	1.983	2.121	2.241	2.347	2.442	2.529

Next if $0 < c < \beta$ then we have the unique positive root $\alpha(c)$ of the equation (9).

$$\alpha(c) = \beta \log(\beta/c).$$

(c) *Poisson distribution*

$$dF(x) = e^{-\theta}\theta^x/(x!)\, dm \qquad (m\text{: counting measure}).$$

We have

$$A(x) = \int_x^\infty (y - x)\, e^{-\theta}\theta^y/(y!)\, dm(y) = \theta P_\theta([x]) - x P_\theta([x] + 1),$$

where $P_\theta(k)$ is the cumulative distribution function

$$P_\theta(k) \equiv \sum_{x=k}^{\infty} e^{-\theta}\theta^x/(x!).$$

Thus, if $c = 0$, we get from (8):

$$\mu_n = \theta P_\theta([\mu_{n-1}]) + \mu_{n-1}\{1 - P_\theta([\mu_{n-1}] + 1)\} \qquad (n = 2, 3, \ldots; \mu_1 = \theta)$$

yielding, for example,

$$\begin{aligned} \mu_2 &= \theta P_\theta([\theta]) + \theta\{1 - P_\theta([\theta] + 1)\} \\ &= \theta(e^{-\theta}\theta^{[\theta]}/[\theta]! + 1). \end{aligned}$$

The numerical values of $A(x) = \theta P_\theta([x]) - xP_\theta([x] + 1)$ are given in Table II for $\theta = 1(1)10$.

III. The Case of Normal Distribution with an Unknown Mean

The optimum stopping problem treated in the previous section can be generalized in two directions. The random variables may have unknown distributions; for example, the normal distribution with unit variance and an unknown mean. We have to extract and accumulate information about the unknown true parameter of the population distribution from succesive observations. The situation will be examined in this section.

The other direction of generalization will be the problem of two populations. We have the two populations, distributions of which are both completely known. We are allowed to make at most N_1 and N_2 successive draws from the two populations respectively, and are requested to get a maximum expected payoff. This problem will be considered in the next section.

In this section we examine in detail the situation in which the random distribution variables have a normal distribution $\mathbf{N}(\theta, 1)$ with unknown mean θ. We assume, however, that we do possess an a priori probability distribution for the value of θ, $\xi(\theta)$.

Let us define:

$f_n(x_1, \ldots, x_{N-n}) =$ expected payoff, obtained using an optimal rule when n stages are left in the future and the observed values were $x_1, \ldots, x_{N-n}$ at the preceding stages,
for $n = 0, 1, \ldots, N-1$.

Our fundamental assumption is the usual one that the new a priori distribution function for θ after m successive observations $x_1, \ldots, x_m$ is given by

$$\xi_{x_1, \ldots, x_m}(\theta) = \frac{\xi(\theta)\phi(x_1 - \theta)\ldots\phi(x_m - \theta)}{\int \xi(\theta)\phi(x_1 - \theta)\ldots\phi(x_m - \theta)\, d\theta} \tag{12}$$

where $\phi(x) \equiv (2\pi)^{-1/2} \exp(-x^2/2)$.

On the basis of this assumption, we obtain the fundamental recurrence relation

$$f_{n+1}(x_1, \ldots, x_{N-n-1}) \tag{13}$$

$$= \max \begin{bmatrix} S\colon\ x_{N-n-1} - (N - n - 1)c \\ C\colon\ \iint f_n(x_1, \ldots, x_{N-n-1}, y)\, \xi_{x_1, \ldots, x_{N-n-1}}(\theta)\phi(y - \theta)\, d\theta\, dy \end{bmatrix}$$

$$(n = 0, 1, \ldots, N - 2)$$

with $f_0(x_1, \ldots, x_N) = x_N - cN$.

It is not difficult to show that we have from these equations

$$f_n(x_1, \ldots, x_{N-n}) = \max\,(x_{N-n}, \mu_n(x_1, \ldots, x_{N-n}) - c) - (N - n)c \tag{14}$$

$$(n = 0, 1, \ldots, N - 1)$$

where the functions $\mu_n(x_1, \ldots, x_{N-n})$ are defined by the recurrence relation

$$\mu_n(x_1, \ldots, x_{N-n}) \tag{15}$$

$$= \iint \max\,(y, \mu_{n-1}(x_1, \ldots, x_{N-n}, y) - c)\xi_{x_1, \ldots, x_{N-n}}(\theta)\phi(y - \theta)\, d\theta\, dy$$

for $n = 2, \ldots, N - 1$ and with

$$\mu_1(x_1, \ldots, x_{N-1}) = \int \theta \xi_{x_1, \ldots, x_{N-1}}(\theta)\, d\theta.$$

The optimal choice, when n stages are left and the observed values at the preceding stages were $x_1, \ldots, x_{N-n}$, is made by the following rule:

$$\begin{cases} \text{if} \quad x_{N-n} > \mu_n(x_1, \ldots, x_{N-n}) - c, \text{ stop sampling;} \\ \text{if otherwise, continue sampling} \end{cases} \tag{16}$$

for $n = 1, \ldots, N - 1$.

For $n = N$ we need another consideration. If we do not enter the game the payoff is of course 0. On the other hand, if we enter the game taking the first observation, the expected payoff will be

$$\iint f_{N-1}(y)\xi(\theta)\phi(y-\theta)\,d\theta\,dy$$

$$= \int \{\max(y, \mu_{N-1}(y) - c) - c\}\,dy \int \xi(\theta)\phi(y-\theta)\,d\theta \qquad \text{(by (14))}$$

$$= \mu_N - c$$

where we have set

$$\mu_N \equiv \iint \max(y, \mu_{N-1}(y) - c)\xi(\theta)\phi(y-\theta)\,d\theta\,dy. \tag{17}$$

Thus we have a rule which determines the optimum initial choice, when any observation is not yet drawn, as follows:

$$\begin{cases} \text{if} \quad 0 > \mu_N - c, & \text{do not enter the game;} \\ \text{if} \quad 0 \leqslant \mu_N - c, & \text{take the first observation} \end{cases} \tag{18}$$

The preceding results are summarized in

THEOREM 2. *The optimum stopping rule for the problem is determined by* (18) *and* (16). *The expected payoff obtained using the optimal rule is* $\max(\mu_N - c, 0)$.

Let us now introduce the two functions

$$A^*(x) \equiv \int \max(y - x, 0)\phi(y)\,dy$$

and

$$K(x) \equiv A^*(-x) - x$$

for later use. Clearly $A^*(x)$ is a special form of $A(x)$ defined by (6) when $dF(y) = \phi(y)\,dy$. It is easily shown that these functions are both strictly decreasing with values ranged from $+\infty$ to 0.

It is to be noted that an interesting and important simplification occurs if the prior distribution for θ is normal. Let us consider the case in which

$$\xi(\theta) = \frac{1}{\sigma}\phi\left(\frac{\theta - \theta_0}{\sigma}\right) \tag{19}$$

and σ is made to tend infinitely large.

Making the substitution (19) in the definition (12) of $\xi_{x_1,\ldots,x_m}(\theta)$, the direct calculation changes it to the form

$$\xi_{x_1,\ldots,x_m}(\theta) = (m+\sigma^{-2})^{1/2}\phi\left\{(m+\sigma^{-2})^{1/2}\left(\theta - \frac{\theta_0\sigma^{-2}+m\bar{x}_m}{\sigma^{-2}+m}\right)\right\}$$

where $\bar{x}_m \equiv (x_1 + \ldots + x_m)/m$. This expression tends to $\sqrt{m}\,\phi(\sqrt{m}(\theta-\bar{x}_m))$ as $\sigma \to \infty$, independently of θ_0. Hence we have, in the limit case of $\sigma \to \infty$, from (15)

$$\mu_1(x_1,\ldots,x_{N-1}) = \bar{x}_{N-1},$$

$$\begin{aligned}
&\mu_2(x_1,\ldots,x_{N-2})\\
&= \iint \max(y, \mu_1(x_1,\ldots,x_{N-2},y) - c)\xi_{x_1,\ldots,x_{N-2}}(\theta)\phi(y-\theta)\,d\theta\\
&= \sqrt{N-2}\int \max\left(y, \frac{N-2}{N-1}\bar{x}_{N-2} + \frac{y}{N-1} - c\right)dy\\
&\qquad\qquad \times \int \phi(\sqrt{N-2}(\theta-\bar{x}_{N-2}))\phi(y-\theta)\,d\theta\\
&= \sqrt{\frac{N-2}{N-1}}\int \max\left(y, \frac{N-2}{N-1}\bar{x}_{N-2} + \frac{y}{N-1} - c\right)\\
&\qquad\qquad \times \phi\left(\frac{y-\bar{x}_{N-2}}{\sqrt{(N-1)/(N-2)}}\right)dy\\
&= \left(\frac{N-2}{N-1}\right)^{3/2}\int \max\left(y - \bar{x}_{N-2} + \frac{N-1}{N-2}c, 0\right)\\
&\qquad\qquad \times \phi\left(\frac{y-\bar{x}_{N-2}}{\sqrt{(N-1)/(N-2)}}\right)dy + \bar{x}_{N-2} - c\\
&= \bar{x}_{N-2} + \sqrt{\frac{N-2}{N-1}}K\left(\sqrt{\frac{N-1}{N-2}}c\right) \equiv \bar{x}_{N-2} + K_{2,N}(c)
\end{aligned}$$

since for every μ and v (> 0) we have

$$\begin{aligned}
v^{-3}\int \max(y-\mu+v^2c, 0)\phi\left(\frac{y-\mu}{v}\right)dy &= v^{-1}\int \max(y+vc, 0)\phi(y)\,dy\\
&= v^{-1}A^*(-vc).
\end{aligned}$$

Similarly, we get, in the limit case of $\sigma \to \infty$,

$$\mu_3(x_1, \ldots, x_{N-3}) = \iint \max(y, \mu_2(x_1, \ldots, x_{N-3}, y) - c)$$

$$\times \xi_{x_1, \ldots, x_{N-3}}(\theta)\phi(y - \theta)\, d\theta\, d$$

$$= \sqrt{N-3} \int \max\left(y, \frac{N-3}{N-2}\bar{x}_{N-3} + \frac{y}{N-2} + K_{2,N}(c) - c\right) d$$

$$\times \int \phi(\sqrt{N-3}(\theta - \bar{x}_{N-3}))\phi(y - \theta)\, d$$

$$= \sqrt{\frac{N-3}{N-2}} \int \max\left(y, \frac{N-3}{N-2}\bar{x}_{N-3} + \frac{y}{N-2} + K_{2,N}(c) - c\right.$$

$$\left.\times \phi\left(\frac{y - \bar{x}_{N-3}}{\sqrt{(N-2)/(N-3)}}\right) d\right.$$

$$= \left(\frac{N-3}{N-2}\right)^{3/2} \int \max\left(y - \bar{x}_{N-3} + \frac{N-2}{N-3}(c - K_{2,N}(c)), 0\right)$$

$$\times \phi\left(\frac{y - \bar{x}_{N-3}}{\sqrt{(N-2)/(N-3)}}\right) dy + \bar{x}_{N-3} - (c - K_{2,N}(c))$$

$$= \sqrt{\frac{N-3}{N-2}}\, K\left[\sqrt{\frac{N-2}{N-3}}(c - K_{2,N}(c))\right] \equiv \bar{x}_{N-3} + K_{3,N}(c).$$

These derivations suggest that perhaps $\mu_n(x_1, \ldots, x_{N-n})$ would always have the form

$$\mu_n(x_1, \ldots, x_{N-n}) = \bar{x}_{N-n} + K_{n,N}(c) \tag{20}$$

where

$$K_{n,N}(c) = \sqrt{\frac{N-n}{N-n+1}}\, K\left[\sqrt{\frac{N-n+1}{N-n}}(c - K_{n-1,N}(c))\right] \tag{21}$$

for $n = 2, \ldots, N-1$ with $K_{1,N}(c) \equiv 0$; and we can immediately establish this by induction.

Now we can state:

COROLLARY 2.1. *Suppose that we must necessarily enter the game. Then the optimum stopping rule for the problem, in the case of normal prior distribution for the unknown mean and in the limit case of infinite variance, is determined by* (16) *with* $\mu_n(x_1, \ldots, x_{N-n})$ *defined by* (20). *The expected* *payoff is then* $\theta - c + \max(K_{N-1,N}(c) - c, 0)$, *where* θ *is the unknown true value of the mean of the normal population.*

PROOF: The first part is obvious by (14) and (16). We have only to show the second part. From (14) and (20) we see

$$\begin{aligned} f_{N-1}(x) &= \max(x, \mu_{N-1}(x) - c) - c \\ &= \max(x, x + K_{N-1,N}(c) - c) - c \\ &= x - c + \max(K_{N-1,N}(c) - c, 0). \end{aligned} \tag{22}$$

Hence the result is immediate.

As an example, if $c = 0$ we have by (21) the strictly increasing sequence

$$K_{n,N}(0) = \sqrt{\frac{N-n}{N-n+1}}\, A^*\left(\sqrt{\frac{N-n+1}{N-n}}\, K_{n-1,N}(0)\right) + K_{n-1,N}(0),$$

$$(n = 2, \ldots, N-1;\ K_{1,N}(0) = 0).$$

When $N = 10$, we get the values:

n	2	3	4	5	6	7	8	9
$K_{n,10}(0)$	0.376	0.591	0.737	0.847	0.930	0.993	1.036	1.058

Stopping regions of the optimum rule are determined by

$$x_m > \bar{x}_m + K_{N-m,N}(0) \qquad (m = 1, \ldots, N-1)$$

and the expected payoff is then $\theta + \max(K_{N-1,N}(0), 0)$.

COROLLARY 2.2. *Suppose that we must necessarily enter the game, and assume that we are allowed to make an infinitely many successive draws from the population. Let* $c > 0$. *Then the optimum stopping rule for the problem, in the case of normal prior distribution for the unknown mean and in the limit case of infinite variance, is determined as follows:*

$$\textit{stop with the first } x_m > \bar{x}_m + K_m(c) - c,$$

where $K_m(c)$ *is defined by the recurrence relation*

$$K_m(c) = \left(\frac{m}{m+1}\right)^{1/2} K\left[\left(\frac{m+1}{m}\right)^{1/2} (c - K_{m+1}(c))\right] \tag{23}$$

for $m = 1, 2, \ldots,$ *with* $K_1(c) \equiv \lim_{N\to\infty} K_{N-1,N}(c)$. *The expected payoff is then* $\theta - c + \max(K_1(c) - c, 0)$.

PROOF: From (20) and (21), we have for every fixed m

$$\mu_{N-m}(x_1, \ldots, x_m) = \bar{x}_m + K_{N-m,N}(c)$$

$$K_{N-m,N}(c) = \left(\frac{m}{m+1}\right)^{1/2} K\left[\left(\frac{m+1}{m}\right)^{1/2} (c - K_{N-m-1,N}(c))\right].$$

Hence $K_m(c) \equiv \lim\limits_{N\to\infty} K_{N-m,N}(c)$ exists for every finite m and satisfies (23), provided $K_1(c)$ exists.

The author cannot prove convergence of $K_{N-1,N}(c)$ $(N \to \infty)$. But if we can assume the convergence, then we can state the above corollary.

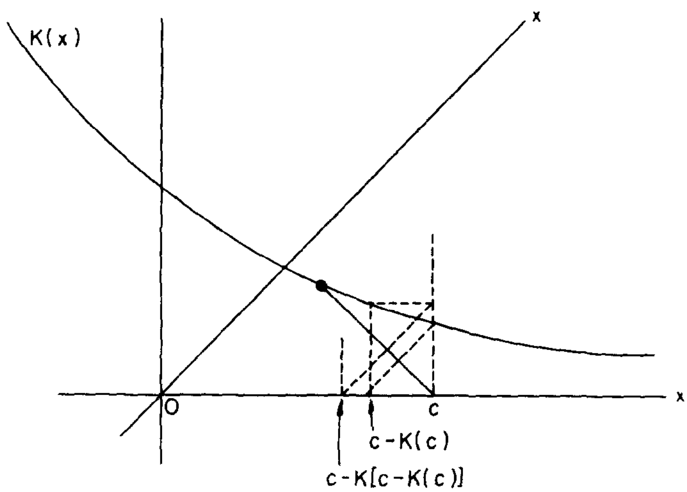

FIG. 2

Now we can easily see (as in Fig. 2) convergence of $\{\xi_n\}$ defined by $\xi_{n+1} = K(c - \xi_n)$ where $K(x) \equiv A^*(-x) - x$. The limit of convergence is given by $\lim\limits_{n\to\infty} \xi_n = \alpha^*(c) + c$, where $\alpha^*(c)$ is the unique root of the equation

$$A^*(\alpha^*) \equiv \int_{\alpha^*}^{\infty} (x - \alpha^*)\phi(x)\,dx = c. \tag{24}$$

It is interesting to note that (1) the $\lim\limits_{N\to\infty} K_{n,N}(c)$ exists for every finite n and $\lim\limits_{n\to\infty} \lim\limits_{N\to\infty} K_{n,N}(c) = \alpha^*(c) + c$; (2) if $\lim\limits_{m\to\infty} K_m(c)$ exists, the limit is equal to $\alpha^*(c) + c$.

IV. Optimum Design for the Problem of Two Populations with Known Distributions

As previously stated at the beginning of the preceding section, we shall discuss here the case of two populations Π_1 and Π_2, distributions of which are both completely known. We can make at most N_1 and N_2 successive draws from Π_1 and Π_2 respectively. We observe the random variables from Π_1 or from Π_2 sequentially and can stop at any time. If we stop with $x_n(y_n)$ from $\Pi_1(\Pi_2)$ after $n = n_1 + n_2$ observations consisting of n_1 drawings from Π_1 and n_2 drawings from Π_2, then we receive the payoff

$$x_n - c_1 n_1 - c_2 n_2 \qquad (y_n - c_1 n_1 - c_2 n_2).$$

We are requested to obtain a maximum expected payoff. The situation therefore is that after each observation we must decide whether to continue sampling or not, and if we decide to continue we must decide which population we should sample next.

Let us define:

$f_{n_1, n_2}(x)$ = expected payoff, obtained using an optimal rule when there remain n_1 drawings from Π_1 and n_2 drawings from Π_2 and starting with the observed value x just drawn at the previous stage for $n_i = 0, 1, \ldots, N_i$ $(i = 1, 2)$, but with $n_1 n_2 \neq N_1 N_2$.

Then we at once get

$$\begin{aligned}
f_{n_1+1, n_2+1}(x) &= \max \begin{bmatrix} S: & x - (N_1 - n_1 - 1)c_1 - (N_2 - n_2 - 1)c_2 \\ C_1: & \int f_{n_1, n_2+1}(u)\, dF(u) \\ C_2: & \int f_{n_1+1, n_2}(v)\, dG(v) \end{bmatrix} \\
f_{n_1+1, 0}(x) &= \max \begin{bmatrix} S: & x - (N - n_1 - 1)c_1 - N_2 c_2 \\ C_1: & \int f_{n_1, 0}(u)\, dF(u) \end{bmatrix} \\
f_{0, n_2+1}(x) &= \max \begin{bmatrix} S: & x - N_1 c_1 - (N_2 - n_2 - 1)c_2 \\ C_2: & \int f_{0, n_2}(v)\, dG(v) \end{bmatrix}
\end{aligned} \tag{25}$$

$$\begin{pmatrix} n_1 = 0, 1, \ldots, N_1 - 1;\ n_2 = 0, 1, \ldots, N_2 - 1 \\ \text{with } n_1 n_2 \neq (N_1 - 1)(N_2 - 1) \text{ in } (25) \end{pmatrix}$$

$$f_{0,0}(x) = x - N_1 c_1 - N_2 c_2$$

where $F(u)$ and $G(v)$ are the cumulative distribution functions of Π_1 and Π_2 respectively.

From the second and third recurrence relations we obtain

$$\left.\begin{aligned} f_{n,0}(x) &= \max(x, \mu_n^{(1)} - c_1) - (N_1 - n)c_1 - N_2 c_2 \\ f_{0,n}(x) &= \max(x, \mu_n^{(2)} - c_2) - N_1 c_1 - (N_2 - n)c_2 \end{aligned}\right\} \tag{26}$$

respectively, where the sequences $\{\mu_n^{(1)}\}$ and $\{\mu_n^{(2)}\}$ are defined by

$$\begin{aligned} \mu_n^{(1)} &= \int \max(x, \mu_{n-1}^{(1)} - c_1)\, dF(x), \qquad (n = 1, \ldots, N_1;\ \mu_0^{(1)} = -\infty) \\ \mu_n^{(2)} &= \int \max(y, \mu_{n-1}^{(2)} - c_2)\, dG(y), \qquad (n = 1, \ldots, N_2;\ \mu_0^{(2)} = -\infty). \end{aligned} \tag{27}$$

Combining these with the first fundamental recurrence relation (24) we can get $f_{n_1, n_2}(x)$ for every $n_i = 0, 1, \ldots, N_i$ $(i = 1, 2)$, except for $n_1 n_2 = N_1 N_2$. And the corresponding optimal choices are determined.

A consequence of these facts is given by

THEOREM 3. *The expected payoff, for the problem of two populations, obtained using the optimum stopping rule is*

$$\max\left[0, \quad \int f_{N_1 - 1, N_2}(u)\, dF(u), \quad \int f_{N_1, N_2 - 1}(v)\, dG(v)\right] \tag{28}$$

where $f_{N_1-1,N_2}(x)$ *and* $f_{N_1,N_2-1}(x)$ *are derived from* (25) *and* (26).

PROOF. The expected payoff f_{N_1,N_2} when using the optimal rule is evidently equal to

$$f_{N_1,N_2} = \max \begin{bmatrix} S: & 0 \\ C_1: & \int f_{N_1-1,N_2}(u)\, dF(u) \\ C_2: & \int f_{N_1,N_2-1}(v)\, dG(v) \end{bmatrix}$$

where the symbols S and C_i $(i = 1, 2)$ denote the actions "stop sampling" and "continue sampling from Π_i" respectively.

An example in which the distributions are both normal is shown in the following and derivation of the optimal rules for several cases of various parameter values will be worked out numerically.

Let

$$N_1 = N_2 = 2, \qquad c_1 = c_2 = 0$$

$$dF(x) = \frac{1}{\sigma_1}\phi\left(\frac{x-\theta_1}{\sigma_1}\right)dx, \qquad dG(y) = \frac{1}{\sigma_2}\phi\left(\frac{y-\theta_2}{\sigma_2}\,dy\right)$$

then we get from (27) and (26)

$$\mu_1^{(i)} = \theta_i$$
$$\mu_2^{(i)} = \theta_i + \sigma_i\,\phi(0) \qquad (i = 1, 2)$$

and

$$f_{10}(x) = \max(x, \theta_1)$$
$$f_{20}(x) = \max(x, \theta_1 + \sigma_1\,\phi(0))$$
$$f_{01}(x) = \max(x, \theta_2)$$
$$f_{02}(x) = \max(x, \theta_2 + \sigma_2\,\phi(0))$$

respectively. Substituting these in (25) we obtain

$$f_{11}(x) = \max(x, \omega_{11})$$
$$f_{12}(x) = \max(x, \omega_{12})$$
$$f_{21}(x) = \max(x, \omega_{21})$$

where

$$\omega_{11} \equiv \max\begin{bmatrix} C_1\colon\ \theta_2 + \sigma_1 A^*\left(\dfrac{\theta_2-\theta_1}{\sigma_1}\right) \\ C_2\colon\ \theta_1 + \sigma_2 A^*\left(\dfrac{\theta_1-\theta_2}{\sigma_2}\right)\end{bmatrix}$$

$$\omega_{12} \equiv \max\begin{bmatrix} C_1\colon\ \theta_2 + \sigma_2\,\phi(0) + \sigma_1 A^*\left(\dfrac{\theta_2+\sigma_2\,\phi(0)-\theta_1}{\sigma_1}\right) \\ C_2\colon\ \omega_{11} + \sigma_2 A^*\left(\dfrac{\omega_{11}-\theta_2}{\sigma_2}\right)\end{bmatrix}$$

$$\omega_{21} \equiv \max\begin{bmatrix} C_1\colon\ \theta_1 + \sigma_1\,\phi(0) + \sigma_2 A^*\left(\dfrac{\theta_1+\sigma_1\,\phi(0)-\theta_2}{\sigma_2}\right) \\ C_2\colon\ \omega_{11} + \sigma_1 A^*\left(\dfrac{\omega_{11}-\theta_1}{\sigma_1}\right)\end{bmatrix}$$

from which we finally get by (28) the expected payoff

$$f_{22} = \max(0, \omega_{22})$$

where

$$\omega_{22} \equiv \max \left| \begin{array}{l} C_1:\ \omega_{12} + \sigma_1 A^*\left(\dfrac{\omega_{12} - \theta_1}{\sigma_1}\right) \\ \\ C_2:\ \omega_{21} + \sigma_2 A^*\left(\dfrac{\omega_{21} - \theta_2}{\sigma_2}\right) \end{array} \right| .$$

The optimum stopping rule will now be clear and we shall omit it.

TABLE I

THE NORMAL DISTRIBUTION $A^*(x) = \int_x^\infty (y - x)\phi(y)\,dy = \phi(x) - x\Phi(x)$ WHERE $\phi(x) \equiv (2\pi)^{-1/2} e^{-x^2/2}$ AND $\Phi(x) = \int_x^\infty \phi(y)\,dy$

	0.08	0.06	0.04	0.02	0.00
− 1.9	1.9890	1.9694	1.9500	1.9305	1.8910
8	1.8917	1.8723	1.8529	1.8336	1.8153
7	1.7950	1.7758	1.7566	1.7374	1.7183
6	1.6992	1.6805	1.6611	1.6421	1.6232
5	1.6044	1.5855	1.5667	1.5480	1.5293
− 1.4	1.5107	1.4921	1.4736	1.4551	1.4367
3	1.4183	1.4000	1.3818	1.3636	1.3455
2	1.3275	1.3095	1.2917	1.2738	1.2561
1	1.2387	1.2209	1.2034	1.1859	1.1686
1.0	1.1514	1.1342	1.1171	1.1002	1.0833
− 0.9	1.0665	1.0499	1.0333	1.0168	1.0004
8	0.9842	0.9680	0.9520	0.9360	0.9202
7	0.9045	0.8889	0.8735	0.8581	0.8429
6	0.8278	0.8128	0.7980	0.7833	0.7687
5	0.7542	0.7399	0.7257	0.7117	0.6978
− 0.4	0.6840	0.6704	0.6549	0.6436	0.6304
3	0.6174	0.6045	0.5918	0.5792	0.5668
2	0.5545	0.5424	0.5304	0.5186	0.5069
1	0.4954	0.4840	0.4728	0.4618	0.4509
0	0.4402	0.4297	0.4193	0.4090	0.3989

	0.00	0.02	0.04	0.06	0.08
0.0	0.3989	0.3892	0.3793	0.3697	0.3602
0.1	0.3509	0.3418	0.3328	0.3240	0.3154
0.2	0.3069	0.2986	0.2904	0.2824	0.2745
0.3	0.2668	0.2592	0.2518	0.2445	0.2374
0.4	0.2304	0.2236	0.2169	0.2104	0.2040
0.5	0.1978	0.1917	0.1851	0.1799	0.1770
0.6	0.1687	0.1633	0.1580	0.1528	0.1478
0.7	0.1429	0.1381	0.1335	0.1289	0.1245
0.8	0.1202	0.1160	0.1120	0.1080	0.1042
0.9	0.1004	0.0968	0.0933	0.0899	0.0865
1.0	0.0833	0.0802	0.0772	0.0742	0.0714
1.1	0.0686	0.0660	0.0634	0.0608	0.0584
1.2	0.0561	0.0538	0.0517	0.0495	0.0475
1.3	0.0455	0.0436	0.0418	0.0400	0.0383
1.4	0.0368	0.0350	0.0336	0.0321	0.0308
1.5	0.0293	0.0280	0.0267	0.0255	0.0244
1.6	0.0232	0.0221	0.0213	0.0202	0.0192
1.7	0.0183	0.0174	0.0166	0.0158	0.0150
1.8	0.0143	0.0136	0.0130	0.0123	0.0116
1.9	0.0111	0.0105	0.0100	0.0095	0.0090
2.0	0.0085	0.0081	0.0076	0.0072	0.0068
2.1	0.0065	0.0061	0.0058	0.0055	0.0052
2.2	0.0049	0.0046	0.0044	0.0041	0.0039
2.3	0.0037	0.0035	0.0033	0.0031	0.0029
2.4	0.0027	0.0026	0.0024	0.0023	0.0021
2.5	0.0020	0.0019	0.0018	0.0017	0.0016
2.6	0.0014	0.0014	0.0013	0.0012	0.0011
2.7	0.0011	0.0010	0.0009	0.0009	0.0008
2.8	0.0008	0.0007	0.0007	0.0006	0.0006
2.9	0.0005	0.0005	0.0005	0.0004	0.0004

TABLE II

THE POISSON DISTRIBUTION $A(x) = \theta P_\theta([x]) - x P_\theta([x] + 1)$,

WHERE $P_\theta(k) = \sum_{x=k}^{\infty} e^{-\theta} \theta^x/(x!)$

$x \backslash \theta$	1	2	3	4	5	6	7	8	9	10
0.1	0.9368	1.9135	2.9050	3.9017	4.9067	5.9006	6.9001	7.9001	8.9001	0.9001
0.2	0.8736	1.8271	2.8096	3.8064	4.8014	5.8005	6.8002	7.8001	8.8000	9.8000
0.3	0.8104	1.7406	2.7144	3.7055	4.7021	5.7007	6.7003	7.7001	8.7000	9.7000
0.4	0.7472	1.6542	2.6200	3.6073	4.6027	5.6010	6.6004	7.6002	8.6000	9.6000
0.5	0.6839	1.5677	2.5249	3.5092	4.5034	5.5013	6.5005	7.5002	8.5001	9.5000
0.6	0.6280	1.4812	2.4929	3.4110	4.4040	5.4015	6.4005	7.4002	8.4001	9.4000
0.7	0.5575	1.3947	2.3349	3.3126	4.3047	5.3018	6.3006	7.3002	8.3001	9.3000
0.8	0.4943	1.3083	2.2388	3.2147	4.2054	5.2020	6.2007	7.2003	8.2001	9.2000
0.9	0.4311	1.2380	2.1448	3.1165	4.1061	5.1022	6.1008	7.1003	8.1001	9.1000
1.0	0.3678	1.1989	2.0497	3.0182	4.0085	5.0025	6.0009	7.0003	8.0002	9.0000
1.2	0.3150	1.0165	1.8896	2.8366	3.8166	4.8060	5.8024	6.8009	7.8004	8.8001
1.4	0.2622	0.9135	1.7294	2.6549	3.6247	4.6094	5.6038	6.6015	7.6006	8.6002
1.6	0.2093	0.7789	1.5692	2.4733	3.4328	4.4129	5.4053	6.4021	7.4009	8.4003
1.8	0.1565	0.6602	1.4091	2.2916	3.2409	4.2164	5.2068	6.2027	7.2011	8.2004
2.0	0.1036	0.5413	1.2490	2.1099	3.0472	4.0198	5.0082	6.0033	7.0014	8.0005
3.0	0.0234	0.2180	0.6721	1.3480	2.1718	3.0818	4.0794	4.9631	6.0076	7.0033
4.0	0.0044	0.0751	0.2194	0.7815	1.4368	2.1330	3.1196	4.0595	5.0288	6.0136
5.0	0.0007	0.0225	0.1346	0.4103	0.8774	1.5125	2.2926	3.1592	4.0838	5.0430
6.0	0.0000	0.0059	0.0507	0.1955	0.4933	0.9837	1.5933	2.3503	3.1995	4.1100
7.0		0.0014	0.0182	0.0848	0.2555	0.5701	1.0430	1.6638	2.4063	3.2402
8.0		0.0003	0.0053	0.0336	0.1221	0.3140	0.6418	1.1167	1.7301	2.4604
9.0		0.0000	0.0015	0.0123	0.0540	0.1623	0.3709	0.7092	1.1858	1.7932
10.0			0.0002	0.0041	0.0222	0.0773	0.2013	0.4259	0.7732	1.2511
11.0			0.0001	0.0013	0.0085	0.0347	0.1028	0.2418	0.4792	0.8342
12.0				0.0004	0.0030	0.0140	0.0495	0.1298	0.2822	0.5310
13.0				0.0001	0.0010	0.0058	0.0225	0.0660	0.1580	0.3224
14.0					0.0003	0.0022	0.0096	0.0319	0.0841	0.1871
15.0					0.0001	0.0008	0.0039	0.0146	0.0427	0.1035
16.0						0.0003	0.0015	0.0064	0.0206	0.0547
17.0						0.0001	0.0006	0.0020	0.0095	0.0277
18.0							0.0002	0.0010	0.0042	0.0134
19.0							0.0001	0.0004	0.0018	0.0062
20.0									0.0007	0.0028

To explain this example more vividly we work out the following four cases numerically (see Table III).

TABLE III

Cases \ Distributions	$F = \mathrm{N}(\theta_1, \sigma_1^2)$	$G = \mathrm{N}(\theta_2, \sigma_2^2)$
(a)	(0,1)	(0,4)
(b)	(0,1)	(1,1)
(c)	(0,1)	(1,4)
(d)	(0,4)	(1,1)

The optimum stopping rules for these four cases are shown in Tables IV and V.

TABLE IV

Case	1st move[a]	2nd move	3rd move	4th move
(a)	Sample Π_2 →	if $y_1 > 1.013$, stop		
		otherwise Π_1 →	if $x_1 > 0.798$, stop	
			otherwise Π_2 →	if $y_2 > 0$, stop
				otherwise Π_1 → stop with x_2
				Expected payoff 1.405
(b)	Sample Π_2 →	if $y_1 > 1.168$, stop		
		otherwise Π_1 →	if $x_1 > 1.083$, stop	
			if otherwise	
			sample Π_1 →	if $x_2 > 1$ stop
				otherwise Π_2 → stop with y_2
			or	
			sample Π_2 →	if $y_2 > 0$, stop
				otherwise Π_1 → stop with x_2
				Expected payoff 1.488

[a] In columns of the 2nd, 3rd, and 4th moves, "if otherwise sample Π" is abbreviated by "otherwise Π_i".

TABLE V

Case	1st move	2nd move	3rd move	4th move	
(c)	Sample Π_1 →	$\begin{cases} \text{if } x_1 > 2.009, \text{ stop} \\ \text{otherwise } \Pi_2 \end{cases}$ →	$\begin{cases} \text{if } y_1 > 1.395, \text{ stop} \\ \text{otherwise } \Pi_2 \end{cases}$ →	$\begin{cases} \text{if } y_2 > 0, \text{ stop} \\ \text{otherwise } \Pi_1 \end{cases}$ →	stop with x_2 Expected payoff 2.017
(d)	Sample Π_1 →	$\begin{cases} \text{if } x_1 > 1.685, \text{ stop} \\ \text{otherwise } \Pi_1 \end{cases}$ →	$\begin{cases} \text{if } x_2 > 1.399, \text{ stop} \\ \text{otherwise } \Pi_2 \end{cases}$ →	$\begin{cases} \text{if } y_2 > 1, \text{ stop} \\ \text{otherwise } \Pi_2 \end{cases}$ →	stop with y_2 Expected payoff 1.907

Let $f_{N_i}^{(i)}$ $(i = 1, 2)$ be the expected payoff when using the optimal rule for the case of the only one population Π_i with the maximum possible permissible number of drawings N_i. Then we have Table VI.

TABLE VI

	$f_{2,2}$	$f_4^{(1)}$	$f_4^{(2)}$	$f_2^{(1)}$	$f_2^{(2)}$
(a)	1.405	0.790	1.580	0.399	0.798
(b)	1.488	0.790	1.790	0.399	1.399
(c)	2.017	0.790	2.580	0.399	1.798
(d)	1.907	1.580	1.790	0.798	1.399

References

1. Bellman, R. "Dynamic Programming." Princeton Univ. Press, 1957.
2. Robbins, H. When to stop. *Ann. Math. Stat.* **31**, 241 (1960) (abstract).
3. Sasieni, M., Yaspan, A., and Friedman, L. "Operations Research: Methods and Problems." Wiley, New York, 1959.